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PART III] *SECTION A* [VOL. 23

NOTE ON A COLLINEATION-GROUP CONNECTED WITH A PLANE QUADRANGLE

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ABSTRACT

The object of this short paper is to reckon with the *group* of plane collineations, which permute, among themselves, the four vertices of a given convex (plane) quadrangle. At the end of the paper there are laconic references to a class of "special" functions, which are *invariant* with respect to the collineation-group. We are not aware whether the problem in its present form has attracted much attention heretofore.

ART. 1.—If X, Y, Z be the three centres of a convex plane quadrangle $ABCD$ (as shewn in Fig. 1), the *projective* or *homogeneous* co-ordinates of the four vertices can be taken as:

$$A(-\alpha, \beta, \gamma),\ B(\alpha, -\beta, \gamma),\ C(\alpha, \beta, -\gamma) \text{ and } D(\alpha, \beta, \gamma),$$

provided that XYZ is chosen as the triangle of reference.

Inasmuch as a plane collineation is uniquely defined, when *four* pairs of corresponding points are assigned (provided, of course, that no three points of either tetrad are collinear), it follows that there exist altogether ${}^4P_4$ or 24 distinct collineations, which, by operating on the four vertices

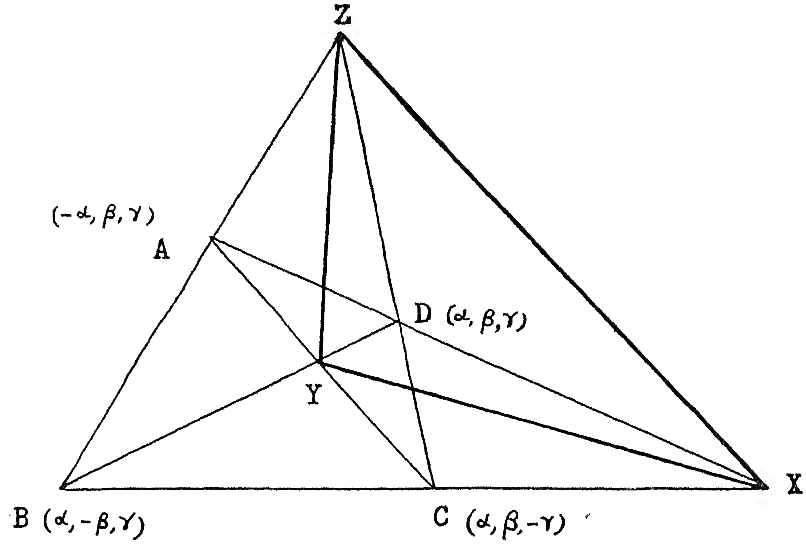

FIG. 1

(A, B, C, D) must produce either a rearrangement of *all* of them, or a rearrangement of *some* of them with invariance of the remaining ones. We shall now find it to our advantage to make free use of the familiar conventions and notations of the Theory of Substitutions. Thus the notation $(ABCD)$ implies that the points A, B, C, D are being converted respectively into the points B, C, D, A, whereas the notation $(ADCB)$ stands for the *inverse* collineation, which converts A, D, C, B into D, C, B, A respectively. Manifestly the collineations:

$$(ABCD),\ (BCDA),\ (CDAB) \text{ and } (DABC)$$

are all identical. Similarly the collineation (ABC) changes A, B, C into B, C, A, whereas the *inverse* collineation (ACB) changes A, C, B into C, B, A. Also the collineation (AB) or (BA) implies that A, B simply change positions. In particular, (AA) means that the point A remains invariant.

For obvious reasons, the product of the two substitutions, *viz.*,

$$(AA)\ (BCD)$$

keeps A fixed and converts B, C, D into C, D, B respectively and the product

$$(AB)\ (CD)$$

implies that A changes place with B and C with D, but the product

$$(AA)\ (BB)\ (CD)$$

keeps both the points A, B fixed and interchanges the points C, D.

With due regard to the aforementioned conventions, we proceed in Art. 2 to study the detailed classification of the 24 collineations, which are connected with the quadrangle $ABCD$ in the manner indicated heretofore.

ART. 2.—Plain reasoning shows that the 24 collineations just talked about generate a *group* G, which can be subdivided into four distinct *classes*, enumerated below:—

CLASS I.—6 collineations (H_1–H_6), in each of which the 4 vertices are rearranged *in a cyclic order*;

CLASS II.—8 collineations (J_1–J_8), in each of which one of the vertices remains *fixed* and the other 3 vertices are re-shuffled *in a cyclic order*;

CLASS III.—6 collineations (K_1–K_6), in each of which two of the vertices remain *fixed* and the other two are interchanged;

CLASS IV.—4 collineations (I, I_1, I_2, I_3), which permit of a *double interchange* of points and also of *repetition* of the same.

It is easy to anticipate beforehand that Class IV contains the '*identity*' element I. The analytic representations of the four *classes* of collineations are appended respectively in the following four Tables I–IV, it being tacitly understood that the ΔXYZ of Art. 1 is chosen as the triangle of reference and that (x, y, z) and (x', y', z') are respectively the initial and final positions of the current point, and that ρ is a factor of proportionality.

TABLE I

(H_1–H_6)

Collineations	Symbolic representations	Analytic representations		
H_1	$(ABDC)$	$\rho x' = x,$	$\rho y' = \frac{\beta}{\gamma} z,$	$\rho z' = -\frac{\gamma}{\beta} y;$
H_2	$(ACDB)$	$\rho x' = x,$	$\rho y' = -\frac{\beta}{\gamma} z,$	$\rho z' = \frac{\gamma}{\beta} y;$
H_3	$(ADCB)$	$\rho x' = \frac{\alpha}{\gamma} z,$	$\rho y' = y,$	$\rho z' = -\frac{\gamma}{\alpha} x;$
H_4	$(ABCD)$	$\rho x' = -\frac{\alpha}{\gamma} z,$	$\rho y' = y,$	$\rho z' = \frac{\gamma}{\alpha} x;$
H_5	$(ADBC)$	$\rho x' = \frac{\alpha}{\beta} y,$	$\rho y' = -\frac{\beta}{\alpha} x,$	$\rho z' = z;$
H_6	$(ACBD)$	$\rho x' = -\frac{\alpha}{\beta} y,$	$\rho y' = \frac{\beta}{\alpha} x,$	$\rho z' = z.$

TABLE II

(J_1–J_8)

Collineations	Symbolic representations	Analytic representations		
J_1	$(AA)(BCD)$	$\rho\frac{x'}{\alpha}=\frac{y}{\beta}$,	$\rho\frac{y'}{\beta}=-\frac{z}{\gamma}$,	$\rho\frac{z'}{\gamma}=\frac{x}{\alpha}$;
J_2	$(AA)(BDC)$	$\rho\frac{x'}{\alpha}=\frac{z}{\gamma}$,	$\rho\frac{y'}{\beta}=\frac{x}{\alpha}$,	$\rho\frac{z'}{\gamma}=-\frac{y}{\beta}$;
J_3	$(BB)(ACD)$	$\rho\frac{x'}{\alpha}=-\frac{z}{\gamma}$,	$\rho\frac{y'}{\beta}=\frac{x}{\alpha}$,	$\rho\frac{z'}{\gamma}=\frac{y}{\beta}$;
J_4	$(BB)(ADC)$	$\rho\frac{x'}{\alpha}=\frac{y}{\beta}$,	$\rho\frac{y'}{\beta}=\frac{z}{\gamma}$,	$\rho\frac{z'}{\gamma}=-\frac{x}{\alpha}$;
J_5	$(CC)(ADB)$	$\rho\frac{x'}{\alpha}=\frac{z}{\gamma}$,	$\rho\frac{y'}{\beta}=-\frac{x}{\alpha}$,	$\rho\frac{z'}{\gamma}=\frac{y}{\beta}$;
J_6	$(CC)(ABD)$	$\rho\frac{x'}{\alpha}=-\frac{y}{\beta}$,	$\rho\frac{y'}{\beta}=\frac{z}{\gamma}$,	$\rho\frac{z'}{\gamma}=\frac{x}{\alpha}$;
J_7	$(DD)(ACB)$	$\rho\frac{x'}{\alpha}=\frac{y}{\beta}$,	$\rho\frac{y'}{\beta}=\frac{z}{\gamma}$,	$\rho\frac{z'}{\gamma}=\frac{x}{\alpha}$;
J_8	$(DD)(ABC)$	$\rho\frac{x'}{\alpha}=\frac{z}{\gamma}$,	$\rho\frac{y'}{\beta}=\frac{x}{\alpha}$,	$\rho\frac{z'}{\gamma}=\frac{y}{\beta}$.

TABLE III

(K_1–K_6)

Collineations	Symbolic representations	Analytic representations		
K_1	$(AA)(DD)(BC)$	$\rho x'=x$,	$\rho y'=\frac{\beta}{\gamma}z$,	$\rho z'=\frac{\gamma}{\beta}y$;
K_2	$(BB)(CC)(AD)$	$\rho x'=-x$,	$\rho y'=\frac{\beta}{\gamma}z$,	$\rho z'=\frac{\gamma}{\beta}y$;
K_3	$(BB)(DD)(CA)$	$\rho x'=\frac{\alpha}{\gamma}z$,	$\rho y'=y$,	$\rho z'=\frac{\gamma}{\alpha}x$;
K_4	$(CC)(AA)(BD)$	$\rho x'=\frac{\alpha}{\gamma}z$,	$\rho y'=-y$,	$\rho z'=\frac{\gamma}{\alpha}x$;
K_5	$(CC)(DD)(AB)$	$\rho x'=\frac{\alpha}{\beta}y$,	$\rho y'=\frac{\beta}{\alpha}x$,	$\rho z'=z$;
K_6	$(AA)(BB)(CD)$	$\rho x'=\frac{\alpha}{\beta}y$,	$\rho y'=\frac{\beta}{\alpha}x$,	$\rho z'=-z$.

TABLE IV

(I, I_1, I_2, I_3)

Collineations	Symbolic representations	Analytic representations		
I (Identity)	$(AA)(BB)(CC)(DD)$	$\rho x' = x,$	$\rho y' = y,$	$\rho z' = z,$
I_1	$(AD)(BC)$	$\rho x' = -x,$	$\rho y' = y,$	$\rho z' = z;$
I_2	$(BD)(CA)$	$\rho x' = x,$	$\rho y' = -y,$	$\rho z' = z;$
I_3	$(CD)(AB)$	$\rho x' = x,$	$\rho y' = y,$	$\rho z' = -z.$

It is palpably plain that the Class IV (consisting of I, I_1, I_2, I_3), is a group in itself, being a sub-group of the bigger group G, but that none of the other three classes is a *group* in the strict sense of the term. The reader, who is so minded, may attempt the arduous task of constructing the "multiplication-table" of the collineation-group G. But without going into details, one can easily verify certain simple relations, *e.g.*—

$$I_1^2 = I_2^2 = I_3^2 = I \quad (= 1),$$
$$H_1^2 = H_2^2 = I_1,$$
$$H_3^2 = H_4^2 = I_2,$$
$$H_5^2 = H_6^2 = I_3,$$

and
$$H_p^4 = 1, \qquad (p = 1, 2, \ldots, 6).$$

ART. 3.—Reverting to the figure of Art. 1, let us keep the three points X, Y, Z *fixed*, and alter the parameters α, β, γ arbitrarily. We then derive an aggregate of ∞^3 of (plane) quadrangles like $ABCD$, each of which has the three (fixed) points X, Y, Z for its centres. If we now take due note of the fact that the anlaytic representations of I, I_1, I_2, I_3 (as recorded in Table IV), are *independent of* α, β, γ, we readily arrive at the following:

PROP.—*If a plane collineation so transforms a quadrangle that its four vertices are interchanged in two distinct pairs, it must operate similarly on an infinitude of other quadrangles.* Further, *all such quadrangles must have the same three 'centres'* (say, X, Y, Z) *and a collineation of the above description must, when referred to XYZ as the "standard triangle" be representable in one or other of the three possible forms*:—

$$\left.\begin{array}{lllll} (I_1) & \ldots & \rho x' = -x, & \rho y' = y, & \rho z' = z; \\ (I_2) & \ldots & \rho x' = x, & \rho y' = -y, & \rho z' = z; \\ (I_3) & \ldots & \rho x' = x, & \rho y' = y, & \rho z' = -z. \end{array}\right\}$$

ART. 4.—Suppose now that $F(\xi\ \eta,\ \zeta)$ is a ternary function, which remains *unaltered* (save as to a multiplicative constant), when any two of the three variables ξ, η, ζ are interchanged. Naturally this assumption implies a number of identical relations of the type:

$$F(\eta, \xi, \zeta) = cF(\xi, \eta, \zeta) \qquad \text{for } \textit{arbitrary } \xi, \eta, \zeta.$$

If we now take a critical survey of the analytical representations of the twenty-four collineations of the group G (as recorded in Tables I–IV), the fact stands out that in each case any one of the three squares:

$$\left(\frac{x}{\alpha}\right)^2, \quad \left(\frac{y}{\beta}\right)^2, \quad \left(\frac{z}{\gamma}\right)^2,$$

is equated to a *definite* numerical multiple of one or other of the three squares:

$$\left(\frac{x'}{\alpha}\right)^2, \quad \left(\frac{y'}{\beta}\right)^2, \quad \left(\frac{z'}{\gamma}\right)^2,$$

(taken in *some* order).

It follows then, as a consequence of the limitation already imposed on $F(\xi, \eta, \zeta)$, that the equation:

$$F\left(\frac{x^2}{\alpha^2}, \frac{y^2}{\beta^2}, \frac{z^2}{\gamma^2}\right) = 0 \tag{A}$$

remains *invariant*, being, as it is, carried over into:

$$F\left(\frac{x'^2}{\alpha^2}, \frac{y'^2}{\beta^2}, \frac{z'^2}{\gamma^2}\right) = 0.$$

The geometrical interpretation is that *every (plane) curve, whose "homogeneous" equation is of the form (A), must remain invariant, when it is submitted to the operation of any one of the* 24 *collineations of the group* G.

We shall now conclude this paper by citing two simple illustrations of the above proposition.

Thus each of the two curves:

$$\frac{x^4}{\alpha^4} + \frac{y^4}{\beta^4} + \frac{z^4}{\gamma^4} = 5\left(\frac{y^2z^2}{\beta^2\gamma^2} + \frac{z^2x^2}{\gamma^2\alpha^2} + \frac{x^2y^2}{\alpha^2\beta^2}\right) \tag{1}$$

and

$$\left(\frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2}\right)\left(\frac{z^2}{\gamma^2} - \frac{x^2}{\alpha^2}\right)\left(\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2}\right) = 0 \tag{2}$$

remains unaffected by the collineations of G.

Manifestly (2) is an improper sextic curve, composed of the three pairs of right lines:

$$(XDA,\ XCB),\ (BYD,\ CYA) \text{ and } (ZDC,\ ZAB), \tag{3}$$

(of Art. 1). By interpreting the collineations geometrically, the reader can easily explain on *á priori grounds* why the six lines, marked (3), must, *when taken together,* remain absolutely invariant, to whichever collineation (of the group G) they are submitted.

By attributing other convenient forms to the ternary quasi-symmetric function $F(\xi, \eta, \zeta)$ of the aforesaid category, the reader can easily derive any number of plane curves like (A), which are, so to say, *automorphic* w.r.t. the collineation-group G.

SOME SELF-RECIPROCAL FUNCTIONS

By Ram Kumar

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[Communicated by Prof. P. L. Srivastava, M.A., D.Phil. (Oxon.), F.N.I., F.N.A.Sc.]

1. The object of this paper is to investigate some self-reciprocal functions with the help of known results that Mohan (1931)

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}(\mu-\nu+1)} \cdot J_{\frac{1}{2}(\mu+\nu)}(xy) \cdot f(y)\, dy \tag{1}$$

and Mohan (1933)

$$g(x) = \int_0^1 \frac{y^{\frac{1}{2}+\mu} \cdot f(xy)\, dy}{(1-y^2)^{1+\frac{1}{2}\mu-\frac{1}{2}\nu}} \tag{2}$$

are R_ν, when $f(x)$ is R_μ.

The R_ν-function 4 (1) obtained in § 4 gives us a generalisation of the well-known result due to Bailey (1930) that

$$x^{\nu-2m-\frac{1}{2}} \cdot e^{\frac{1}{4}x^2} \cdot W_{-\nu+3m-\frac{1}{2},\, m}(\tfrac{1}{2}x^2) \tag{3}$$

is R_ν, which, when expressed in the form of Mac-Robert's function $E(\alpha, \beta :: x)$, can be written as

$$x^{-(\beta+\frac{1}{2})}\, E(\tfrac{1}{2}\nu + \tfrac{1}{2}\beta + \tfrac{1}{2},\, \beta :: \tfrac{1}{2}x^2), \tag{4}$$

where $R(\nu) > -1$, $R(\nu+\beta) > -1$.

2. If we take

$$f(y) = y^{-(\beta+\frac{1}{2})}\, E(\alpha, \beta :: \tfrac{1}{2}y^2),$$

which is $R_{2\alpha-\beta-1}$ (Bailey, 1930), provided $R(\alpha) > 0$ and $R(2\alpha-\beta) > 0$, in the integral formula 1 (1), then

$$g(x) = x^{\alpha-\frac{1}{2}\beta-\frac{1}{2}\nu} \int_0^\infty y^{\alpha-3\beta/2-\frac{1}{2}\nu-\frac{1}{2}} \cdot J_{\alpha-\frac{1}{2}\beta+\frac{1}{2}\nu-\frac{1}{2}}(xy) \cdot E(\alpha,\beta :: \tfrac{1}{2}y^2)\, dy,$$

$$= (\tfrac{1}{2})^{-\alpha+3\beta/2+\frac{1}{2}\nu+\frac{1}{2}} \cdot \frac{\Gamma(\alpha-\beta)\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\beta)} \cdot x^{\beta-\frac{1}{2}}$$

$$\times {}_2F_2\left(\begin{matrix}\alpha, & \beta \\ 3/2-\alpha-\frac{1}{2}\nu+\frac{1}{2}\beta, & \frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\beta\end{matrix};\ \tfrac{1}{2}x^2\right)$$

$$+ (\tfrac{1}{2})^{\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2}} \cdot \frac{\Gamma(\beta-\alpha)\,\Gamma(\alpha)\,\Gamma(2\alpha-\beta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\alpha-\frac{1}{2}\beta)} \cdot x^{2\alpha-\beta-\frac{1}{2}}$$

$$\times\, {}_2F_2\left(\begin{matrix}2\alpha-\beta\;, & \alpha \\ \alpha-\beta+1, & \frac{1}{2}+\frac{1}{2}\nu+\alpha-\frac{1}{2}\beta\end{matrix};\ \tfrac{1}{2}x^2\right),$$

provided $R(2\alpha-\beta)>0$, $R(\alpha)>0$, $R(\nu+3\beta-2\alpha)>-2$.

The above integral has been evaluated with the help of a more general formula given by F. M. Ragab (1950).

Hence, leaving aside the constant factor, we have

$$x^{-(\beta+\frac{1}{2})}\left\{\frac{\Gamma(\alpha-\beta)\,\Gamma(\beta)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\beta+\frac{1}{2})} \cdot \left(\frac{x^2}{2}\right)^{\beta}\right.$$

$$\times\, {}_2\mathrm{F}_2\left(\begin{matrix}\alpha\;, & \beta \\ 3/2-\alpha-\frac{1}{2}\nu+\frac{1}{2}\beta, & \frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\beta\end{matrix};\ \tfrac{1}{2}x^2\right)$$

$$+\frac{\Gamma(\beta-\alpha)\,\Gamma(2\alpha-\beta)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\alpha-\frac{1}{2}\beta)} \cdot \left(\frac{x^2}{2}\right)^{\alpha}$$

$$\left.\times\, {}_2\mathrm{F}_2\left(\begin{matrix}2\alpha-\beta\;, & \alpha \\ \alpha-\beta+1, & \frac{1}{2}+\frac{1}{2}\nu+\alpha-\frac{1}{2}\beta\end{matrix};\ \tfrac{1}{2}x^2\right)\right\} \quad (1)$$

as R_ν, provided $R(\alpha)>0$, $R(2\alpha-\beta)>0$, $R(\nu+3\beta-2\alpha)>-2$.

3. If we put $\alpha=1$ and $\beta=1-\nu$ in 2 (1), we get

$$x^{-(3/2-\nu)}\,E(1-\nu,\ 1::\tfrac{1}{2}x^2),\ -1<R(\nu)<3/2$$

as R_ν which is also a particular case of 1 (4).

4. If we take

$$y^{-(\beta+\frac{1}{2})}\cdot E(\alpha,\ \beta::\tfrac{1}{2}y^2),$$

which is $R_{2\alpha-\beta-1}$, provided $R(\alpha)>0$ and $R(2\alpha-\beta)>0$, as $f(y)$ of the integral formula 1 (2), we get

$$g(x) = x^{-(\beta+\frac{1}{2})}\int_0^1 \frac{y^{2(\alpha-\beta-\frac{1}{2})}\cdot E(\alpha,\ \beta::\frac{1}{2}x^2y^2)}{(1-y^2)^{\frac{1}{2}+\alpha-\frac{1}{2}\beta-\frac{1}{2}\nu}}\,dy$$

$$= \tfrac{1}{2}\cdot\Gamma(\alpha)\,\Gamma(-\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}\nu+\tfrac{1}{2})\cdot x^{-(\beta+\frac{1}{2})}$$

$$\times\left[\frac{\Gamma(\beta-\alpha)\,(2\alpha-\beta)}{\Gamma(\alpha-\frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2})}\cdot\left(\frac{x^2}{2}\right)^{\alpha}\right.$$

$$\times\, {}_2F_2\left(\begin{matrix}\alpha\;, & 2\alpha-\beta \\ \alpha-\beta+1, & \alpha-\frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2}\end{matrix};\ \tfrac{1}{2}x^2\right)$$

$$+ \frac{\Gamma(\alpha-\beta)\,\Gamma(\beta)}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2})} \cdot \left(\frac{x^2}{2}\right)^{\beta}$$

$$\times\, {}_2F_2\left(\begin{matrix}\beta, & \alpha \\ \beta-\alpha+1, & \frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2}\end{matrix}; \frac{1}{2}x^2\right)\Big],$$

provided $R(\alpha) > 0$, $R(2\alpha-\beta) > 0$, $R(\nu+\beta-2\alpha) \geqslant -1$.

Hence, leaving aside the constant factor, we have

$$x^{-(\beta+\frac{1}{2})}\left[\frac{\Gamma(\beta-\alpha)\,\Gamma(2\alpha-\beta)}{\Gamma(\alpha-\frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2})} \cdot \left(\frac{x^2}{2}\right)^{\alpha}\right.$$

$$\times\, {}_2F_2\left(\begin{matrix}\alpha, & 2\alpha-\beta \\ \alpha-\beta+1, & \alpha-\frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2}\end{matrix}; \frac{1}{2}x^2\right)$$

$$+ \frac{\Gamma(\alpha-\beta)\,\Gamma(\beta)}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2})} \cdot \left(\frac{x^2}{2}\right)^{\beta}$$

$$\left.\times\, {}_2F_2\left(\begin{matrix}\beta, & \alpha \\ \beta-\alpha+1, & \frac{1}{2}\beta+\frac{1}{2}\nu+\frac{1}{2}\end{matrix}; \frac{1}{2}x^2\right)\right] \quad (1)$$

as R_ν, provided $R(\alpha) > 0$, $R(2\alpha-\beta) > 0$, $R(\nu+\beta-2\alpha) \geqslant -1$.

5. If we put $\alpha = \frac{1}{2}\nu + \frac{1}{2}\beta + \frac{1}{2}$ in 4 (1), we get

$$x^{-(\beta+\frac{1}{2})} \cdot E(\tfrac{1}{2}\nu + \tfrac{1}{2}\beta + \tfrac{1}{2},\ \beta :: \tfrac{1}{2}x^2)$$

as R_ν, provided $R(\nu) > -1$. $R(\nu+\beta) > -1$, which is Bailey's result 1 (4).

My best thanks are due to Dr. R. S. Varma for his help in the preparation of this paper.

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RENNINGER EFFECT IN ANTHRAQUINONE

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Read on December 28, 1954

(Communicated by Professor K. Banerjee, D.SC., F.N.I.)

Abstract

A satisfactory explanation is given for the formation of the forbidden spots which were observed in the zerolayer [010] Weissenberg photograph of Anthraquinone which are shown to be originated due to Renninger effect.

Introduction

The crystal structure of Anthraquinone, $C_{14}H_8O_2$ was determined by Sen (1945) by the two-dimensional Fourier Synthesis. He previously (Sen, 1940) assigned to the crystal the space group $P21/a$, but in his later work (*Thesis*, Calcutta University) mentioned the observation of a few weak forbidden reflections (*hol* with h odd) (Table I), which led him to ascribe $P2_1$ as the correct space group. These forbidden reflections as well as a few more were observed by the author. A critical re-examination of the space group was made by the author by studying the characteristics of the forbidden spots.

Detection of Renninger Reflections in Anthraquinone

The forbidden reflections mentioned above were found to be much sharper than the normal reflections and moreover were not associated with the usual reflections due to CuKβ, which were observed in the case of equally intense normal ones. These forbidden reflections were suspected, by their nature, to be due to Renninger (1937) reflections of the incident beam by two sets of strong planes and were shown to be so by a study of the reflecting conditions of the reciprocal lattice.

The mechanism of formation of the double reflections can be understood by constructing the reciprocal lattice for this crystal. As an example we shall consider the forbidden reflection (302) in some details.

The reciprocal lattice for the equatorial layer of [010] is shown in Fig. 1. This is drawn on the basis of a unit cell dimensions, $a = 15 \cdot 77$ A.U., $b = 3 \cdot 98$ A.U., $C = 7 \cdot 85$ A.U. and $\beta = 102° 43'$ which are the revised ones by the author from a consideration of the high angle spots (for which $\alpha_1\alpha_2$

doublet is well resolved) of the equatorial layer line [010] setting of the Weissenberg photograph.

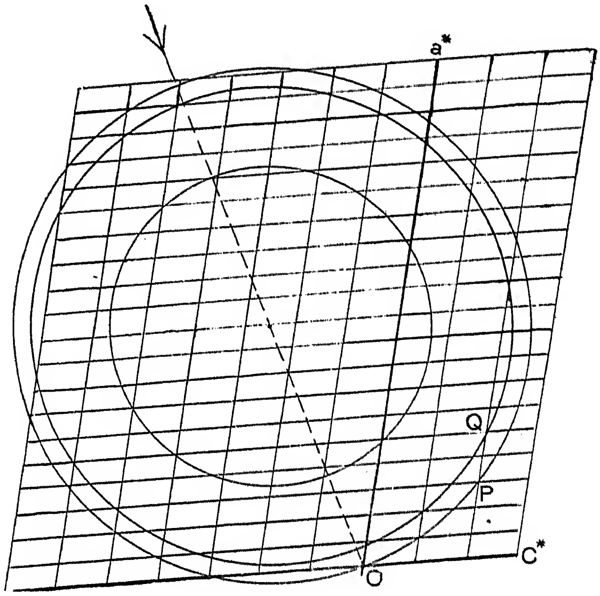

FIG. 1

The circles of reflection for different layers projected on the equatorial layer were drawn with their respective radii on a transparent film with a common diameter to represent the incident beam, traces of which are shown in the diagram. One of the junctions of the diameter with the equatorial layer circle of reflection is the origin of the reciprocal lattice and the film rotated until the circle corresponding to the equatorial layer passes through the point (3, 2). In this position it is observed that the circle of reflection of the 1st layer is passing through points (5, 2) and (2, 1) indicating that the (512) and (211) directions may become probable incident directions for planes whose reciprocal vectors are the vector differences of these two vectors from that of (502) planes to give rise to the reflection (302). Since this graphical method cannot be expected to give a high degree of accuracy for these coincidences, the following analytical method was used to test the accuracy of the results.

It is convenient to express the required results in polar co-ordinates. The equatorial circle is of unit radius and always passes through the origin. It is also passing through a point $P(r, \theta)$. Let the co-ordinates of the centre of the circle be $(1, \alpha)$. Then the equation to the circle is,

$$r/2 = \cos(\alpha - \theta) \text{ (Fig. 2).}$$

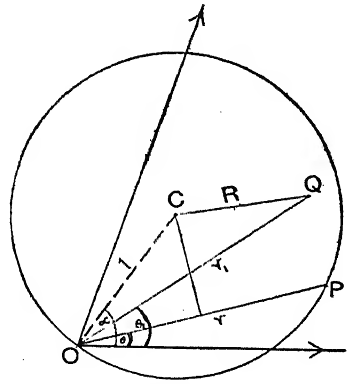

FIG. 2

from which,

$$\alpha = \theta + \cos^{-1} r/2 \tag{1}$$

Let Q be any point (r_1, θ_1), the distance of which from C is R. Then from ΔCOQ,

$$CQ^2 = OC^2 + OQ^2 - 2 \cdot OC \cdot OQ \cos(\alpha - \theta_1)$$

$$\text{i.e.,} \quad R^2 = 1 + r_1{}^2 - 2r_1 \cos(\alpha - \theta_1).$$

The polar co-ordinates of the various points concerned can be obtained from reciprocal lattice, Fig. 1. Polar co-ordinates of the point (302) are given by

$$r = \cdot 5529 \text{ and } \theta = 32°,$$

which leads to $\alpha = 105° \, 57'$.

For (512) with reciprocal lattice co-ordinates (5, 2) and polar co-ordinates (r_1, θ_1) in the plane of the first layer, we have

$$r_1 = \cdot 7082, \; \theta_1 = 43° \, 34'$$

which results in $R = \cdot 919$.

For (211)—reciprocal lattice co-ordinates (2, 1).

Following just the same steps as above R comes to $\cdot 926$.

Both these values for R are in quite good agreement with the radius of the 1st layer line circle of reflection $R = \cos \nu = \cdot 922$, Fig. 3.

If (hkl) are the indices of the forbidden reflection and $h_1k_1l_1$ and $h_2k_2l_2$ are respectively those of the two normal reflections,

$$h = h_1 + h_2, \; k = k_1 + k_2 \text{ and } l = l_1 + l_2.$$

$h_2k_2l_2$ in the case of (512) and (211) are thus found to be $(\overline{2}\,\overline{1}\,0)$ and $(1\,\overline{1}\,1)$ respectively. Each of these reflections of both the pairs is fairly strong.

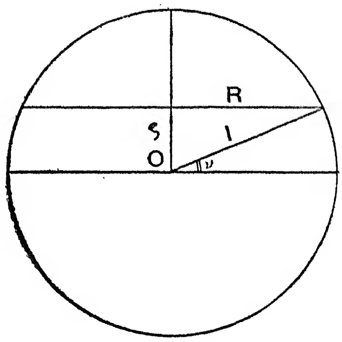

FIG. 3

For each forbidden reflection that was observed a similar procedure was adopted and the pairs from which the contributions were made in giving rise to the particular reflection were brought out. Intensities of these forbidden spots were found to be in reasonable agreement with the expected estimates from their contributors. A list of the forbidden reflections along with their contributors are given in Table I.

TABLE 1

Observed forbidden reflection	Pairs of planes giving rise to the corresponding forbidden reflection
(101)	$(11\bar{3}.\ 0\bar{1}4).\ (211.\ \bar{1}\,\bar{1}0).\ (52\bar{4}\ \bar{4}\bar{2}5)$
(102)	$(212.\ \bar{1}\,\bar{1}0).\ (11\bar{1},\ 0\bar{1}3),\ (413,\ \bar{3}\,\bar{1}\bar{1})$
$(\bar{1}02)$	$(321,\ \bar{4}\bar{2}1),\ (013),\ \bar{1}\,\bar{1}\,\bar{1})$
(302)	$(512,\ \bar{2}\,\bar{1}0),\ (211,\ \bar{1}\bar{1}1)$
$(\bar{5}02)$	$(\bar{2}11,\ \bar{3}\,\bar{1}1),\ (210,\ \bar{7}\,\bar{1}2)$
(103)	$(112,\ 0\bar{1}1,\ (61\bar{4},\ \bar{5}\,\bar{1}7)$
$(\bar{3}03)$	$(110,\ \bar{4}\bar{1}3)$

RESULTS

Since these forbidden reflections are satisfactorily explained, the space-group of Anthraquinone is established as $P2_1/a$. The existence of the centre of symmetry required by $P2_1/a$ was confirmed by the author by a statistical

consideration of the intensity distribution in Anthraquinone and was published elsewhere (1955).

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